Introduction to sea ice modelling

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thermodynamics:
strong heat fluxes over leads

dynamics: ridges and leads, rubble, pack
“Dynamic” duo for Sea Ice

**Dynamics**
- Fast
- Rough

**Thermodynamics**
- Slow
- Smooth

Perovich, 2012, FAMOS
Overview

• **(very short) Introduction:** Sea ice in the climate system

**Thermodynamics**
- heat balance
- heat capacity
- zero-layer, multi-layer models
- salinity, brine, enthalpy
- Snow on ice
- advection
- ice thickness distribution

**Dynamics**
- continuum assumption
- momentum equations
- surface stress
- divergence of internal stress
- rheology, isotropy, anisotropy, Viscous-Plastic, Maxwell-Elasto-Brittle, Mohr-Coulomb
- ice thickness distribution and ridging

**Numerical models**
- solution techniques specific to sea-ice models
- implicit solvers
  - Picard/fixed point/FGMRES, JFNK
- explicit solvers:
  - EVP, mEVP, aEVP
  - EAP
- new rheologies
- Discrete Element Models

• Sea ice models in ECCO -> Ian’s talk

• Biogeochemistry in sea ice models
Sea ice model equations

- Dynamics (momentum equations)
- Advection of sea ice (thickness, concentration, salinity, other tracers)
- Thermodynamics (heat balance, albedo, ice growth, melt, sub-grid ice thickness distributions, melt-ponds, …): $S_{h}, S_{s}, S_{c}$
- Redistribution (Ridging)

For a review of sea ice modeling: Hunke et al. (2010), Lemieux et al. (2017) (book chapter)

Flowchart stolen from https://nsidc.org/cryosphere/seaice/study/modeling.html
thermodynamics: heat balance

- ice-enthalpy includes heat and chemical potential (salinity)

\[ \rho \frac{\partial}{\partial t} E = \frac{\partial}{\partial z} \left( \kappa \frac{\partial T}{\partial t} \right) + Q, \quad E(S, T) = c_{p,i}(T + \mu S) - \Lambda (1 - \phi) - c_{p,w} \mu S \]

- often simply:

\[ E = c_p T \]

- with boundary conditions

  **top:**

  \[ \kappa \frac{\partial T}{\partial z} = Q_T \downarrow \]

  **bottom:**

  \[ T = T_f = T_f(S) \]

  \[ \rho \Lambda \frac{\partial H}{\partial t} = - \kappa \frac{\partial T}{\partial z} \bigg|_{\text{bottom}} - Q_w \uparrow \]
Stefan’s law of ice growth (following Leppäranta, 1993)

- assumptions:
  - no thermal inertial: \( E = 0 \)
  - no internal heat source: \( Q = 0 \)
  - no heat flux from ocean: \( Q_w = 0 \)
  - known surface temperature \( T_0 = T_0(t) \)

\[ \frac{\partial T}{\partial z} = \text{constant} \]

\[ \rho \Lambda \frac{dH}{dt} = -\kappa \frac{\partial T}{\partial z} = \kappa \frac{T_f - T_0(t)}{H} \]

\[ H \frac{dH}{dt} = \frac{1}{2} \frac{dH^2}{dt} dt = \frac{\kappa}{\rho \Lambda} \left[ T_f - T_0(t) \right] \]

\[ \int_0^t \frac{dH^2}{dt} dt' = H(t)^2 - H_0^2 = \frac{2\kappa}{\rho \Lambda} \int_0^t \left[ T_f - T_0(t') \right] dt' \]

\[ H(t) = \sqrt{H_0^2 + \frac{2\kappa}{\rho \Lambda} \int_0^t \left[ T_f - T_0(t') \right] dt'} \]

sum of negative degree days

gives typically 140cm for 180 days of -10K freezing conditions
Sea ice model equations: “0-layer thermodynamics”

\[
\frac{\partial E}{\partial t} = \rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial z} \left( \kappa \frac{\partial T}{\partial t} \right) + Q
\]

- no internal heat source: \( Q = 0 \)
- no thermal inertia \( \Rightarrow \) instantaneous temperature adjustment

\[
c_p = 0 \Rightarrow \frac{\partial T}{\partial z} = \text{constant} = \frac{T_0 - T_f}{h}
\]

- only surface boundary conditions remain: \( \frac{\kappa}{h} (T_0 - T_f) = Q_T \downarrow \)
- with

\[
Q_T \downarrow = Q_{SW\downarrow}(1 - \alpha) + \epsilon Q_{LW\downarrow} - Q_{LW\uparrow}(T_0) + Q_{LH}(T_0) + Q_{SH}(T_0)
\]

\[
\Rightarrow T_0 \Rightarrow Q_T(T_0) \Rightarrow \rho \Lambda \frac{\partial H}{\partial t} = \kappa \frac{T_f - T_0}{H} - Q_w
\]

(Semptner, 1976, 1984)
Sea ice albedo for $Q_{SW \downarrow}(1 - \alpha)$

- simple parameterisations with albedo for ice and snow in freezing (brighter) or melting (darker) conditions
- use melt-pond physics to estimate albedo (e.g., Taylor and Feltham, 2004, Flocco and Feltham, 2007)
- effects of ageing snow and ice, multiple-scattering, absorptive effects of inclusions such as dust and algae (biogeochemistry!)
- important tuning parameter
Snow on sea ice

- snow is a very good insulator:
  - changes surface albedo
  - usually limits shortwave penetration
  - changes vertical diffusion of temperature (from continuity at interface, e.g. Leppäranta, 1991) in 0-layer model:

\[
\kappa_{\text{eff}} = \frac{\kappa_{\text{ice}} \kappa_{\text{snow}}}{\kappa_{\text{snow}} h_{\text{ice}} + \kappa_{\text{ice}} h_{\text{snow}}}
\]

- snow-ice: refreezing of flooded snow on sea ice, especially in Antarctica
going back to more general thermodynamics

- ice-enthalpy
  \[ \rho \frac{\partial}{\partial t} E = \frac{\partial}{\partial z} \left( \kappa \frac{\partial T}{\partial t} \right) + Q \]

- Simplest model is a three-layer model (two ice layers, one snow layer, e.g. Winton, 2000), more layers (Bitz and Lipscomb, 1999) used CICE/ICEPACK.

- any excess conductive heat flux \((k \frac{\partial T}{\partial z})\) through the ice leads to freezing or melting: change of volume \(S_h (=dh/dt)\)

- Hibler (1979): lateral freezing and melting
  \[ S_c = \frac{1 - c}{h_0} \max \left( S_h, 0 \right) + \frac{c}{2h} \min \left( S_h, 0 \right) \]

  lead closing parameter
ice thickness distribution (static)

- thick ice (especially with a snow layer) is a good insulator and limits new ice growth
- simple parameterization scales distribution by mean thickness to always allow thin ice

\[ Q = (1 - c) Q_0 + c \sum_{k=1}^{N} Q \left( \frac{2k - 1}{N} \frac{h}{c} \right) \]

![Graph showing ice thickness distribution](image)

images: Castro-Morales et al. (2014)
dynamic ice thickness distribution: redistribution + ridging

- ice concentration equation is replaced by an equation for thickness distribution function \( g(h) \)

\[
\frac{\partial c}{\partial t} = -\nabla \cdot (c \mathbf{u}) + S_c \quad \rightarrow \quad \frac{\partial g}{\partial t} = -\nabla \cdot (\mathbf{u}g) - \frac{\partial}{\partial h}(fg) + \Psi
\]

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**Figure 4.** Semi-logarithmic plot of average ice draft \( h \) or ice and snow thickness \( h_t \) against probability density in each category for three regional ITD. Blue crosses for model values, red lines for observations. The dashed black lines indicate exponential fits to the model results.

**Figure 5.** Example of variability in ITDs on small local scales. Plotted are ITDs from 50km submarine track segments (red line) with a snapshot from the nearest grid cell (blue bars). All five observations are taken in Fram Strait in spring (S5).

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From Ungermann and Losch (2018)
(RGPS data near SHEBA drift station, 1997, R. Kwok)
Sea Ice Deformation

- ice compression and shear: ridges, rubble fields
- breaking ice: cracks, leads
Importance of sea ice deformations

- Affect the thickness distribution through formation of ridges and leads.
- Heat flux through new leads is 1-2 orders of mag higher than over thick ice (Maykut, 1978)
- 25-40% of new ice formation occurs in leads (Kwok, 2006)
- Ridges affect the air-ice and ice-ocean drag
- Ocean upwelling associated to shear

McPhee et al., 2005
Digression: Newtonian Fluid

1. Fluid is continuous + stresses are a linear function of the strain rates
2. Fluid is isotropic
3. In the limit where the strain rates go to zero, the stresses must reduce to the hydrostatic pressure

\[ \rho \frac{du}{dt} = \nabla \cdot \sigma + R, \quad R = \text{other terms} \]

for PE (ocean): \( \sigma = \nu \nabla \mathbf{u} - p \Rightarrow \nabla \cdot \sigma = \nabla \left( \nu \nabla \mathbf{u} - p \right) \)

strain rates \( (\nabla \mathbf{u})_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \)
Sea ice is different (AIDJEX model)

- ensemble of many ice floes with interactions
- continuity assumption at large scales (questionable)
- non-Newtonian (non-normal) fluid (honey vs. mayonnaise)
- strong in compression, weak in tension, intermediate in shear
- elastic response to small perturbations, plastic to large pert.
- \( \rightarrow \) plastic-elastic model (Coon 1974)
- viscous-plastic framework is similar, but numerically simpler (Hibler 1977, 1979), small strain rates lead to viscous creep.
- granular material (like sand: floes = grains) \( \rightarrow \) Mohr-Coulomb law of failure relates shear to normal stress: \( \tau = \mu \sigma + c \)
Stress tensor

- assumption: symmetric  \[ \Rightarrow \sigma_{21} = \sigma_{12} \]
- Mohr-Coulomb law: \[ \tau = \mu \sigma + c \quad \text{or} \quad \sigma_{II} = \mu \sigma_I + c/2 \]

\[
\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} \Rightarrow \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}
\]

Principle stresses

\[
\sigma_{1,2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}
\]

Normal stress \( \sigma \)

\[
\sigma_I = \frac{1}{2}(\sigma_1 + \sigma_2) = \frac{1}{2}(\sigma_{11} + \sigma_{22})
\]

Shear stress \( \tau \)

\[
\sigma_{II} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2}
\]
principal stress plane and yield curve, plastic limit

\[ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \Rightarrow \sigma_I \text{ and } \sigma_{II} \]

elliptical yield curve: \( F = \left( \frac{\sigma_I + P/2}{P/2} \right)^2 + \left( \frac{\sigma_{II}}{P/(2e)} \right)^2 - 1 = 0 \)

with normal flow rule: \( \dot{\epsilon}_{ij} = \lambda \frac{\partial F(\sigma)}{\partial \sigma_{ij}} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \)

\[ \Rightarrow \text{constitutive relation: } \sigma_{ij} = 2\eta \dot{\epsilon}_{ij} + \delta_{ij} \left[ \zeta - \eta \right] \dot{\epsilon}_{kk} - \frac{P}{2} \]

\[ e = \frac{a}{b}, \quad \zeta = \frac{P}{2\Delta}, \quad \eta = \frac{\zeta}{e^2}, \quad \Delta = \sqrt{(\dot{\epsilon}_{11} + \dot{\epsilon}_{22})^2 + e^{-2}[(\dot{\epsilon}_{11} - \dot{\epsilon}_{22})^2 + 4\dot{\epsilon}_{12}^2]} \]
consequence: sea ice dynamics are very non-linear

\[ m \frac{\partial u}{\partial t} = \nabla \cdot \sigma + R, \quad R = \text{other terms} \]

with \( \sigma_{ij} = \frac{P}{2\Delta} \left\{ 2\dot{e}_{ij} e^{-2} + [(1 - e^{-2})(\dot{e}_{11} + \dot{e}_{22}) - \Delta] \delta_{ij} \right\} \)

with abbreviations

\[ \Delta = \sqrt{(\dot{e}_{11} + \dot{e}_{22})^2 + e^{-2} \left[ (\dot{e}_{11} - \dot{e}_{22})^2 + 4\dot{e}_{12} \right]} \]

\[ \dot{e}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (\text{strain rates}) \]

\[ \rightarrow \quad m \frac{\partial u}{\partial t} \propto \frac{\partial}{\partial x_i} \left( \frac{P}{\Delta} \frac{\partial u_i}{\partial x_j} \right) + \text{similar terms} \]

ice strength parametrizations:

Hibler (1979): \( P = P^* h e^{-C^*(1-c)} \)

Rothrock (1975): \( P = C_f C_p \int_0^\infty h^2 \omega_r(h) dh \)
Sea ice dynamics and solvers

- Picard solvers (LSR, Krylov)
- JFNK solver
- EVP solvers: mEVP, aEVP

- new MEB rheology

- discrete element models (DEM)
solution techniques: Picard method

\[ A(u) \cdot u = b \]
\[ \Rightarrow \text{solve} \quad A(u_{n-1}) \cdot u_n = b \]

• traditional method, e.g., PSOR, Hibler (1979), LSOR, Zhang and Hibler (1997), (Gauss-Seidel) for linear solver
• Krylov method for linear solver (Lemieux and Tremblay, 2009), requires preconditioner
• stable, but slow
solution techniques: JFNK solver

\[ F(u) = A(u) \cdot u - b \]

\[ F(u_n) = F(u_{n-1}) + F'\bigg|_{u_{n-1}} \delta u \overset{!}{=} 0 \]

\[ \Rightarrow \text{solve } \quad F'_n \delta u = -F(u_{n-1}) \quad \Rightarrow \quad u_n = u_{n-1} + \delta u \]

• better (quadratic) convergence near minimum (Lemieux et al. 2010, 2012, Losch et al 2014)
• preconditioner for Krylov solver necessary
• expensive
• unstable, especially at high resolution
• stabilization (e.g. Mehlmann and Richter 2017, involves mixing JFNK and Picard methods)
Picard vs. JFNK

Fig. 4 illustrates what we mean by sharp solution structures. It shows the shear deformation field on 7 January 1990 08Z simulated by the JFNK solver when using the 10-km resolution model and a $c_{nl}$ of 0.001. The shear deformation (second strain rate invariant) is given by

$$
\sqrt{\frac{\partial u}{\partial x}/c_{16}/c_{17} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}/c_{16}/c_{17}^2}
$$

As in Maslowski and Lipscomb [28], who used a model with about the same spatial resolution (9 km), our model simulates basin scale linear kinematic features that resemble the observed ones [29]. Note that the existence of these strong velocity gradients is physically based (VP rheology) and is not a consequence of residual errors in the velocity field approximate solution.

Note that for the JFNK solver, the computational efficiency and failure rate depend on the chosen value of $res$ (Eq. (25)) and that some tuning might slightly modify these results. A larger $res$ tends to increase the computational efficiency and the failure rate.

The lack of convergence (failures) of the JFNK solver and the standard solver is a global convergence issue. When the initial iterate is ''sufficiently close'' to the solution, the solvers always converge. The quality of the initial iterate is determined by the time step compared to the forcing time scale and to the level of convergence of the previous time step solution. A 1-month integration at 40-km resolution with a 1-minute time step (44,640 time steps) for $c_{nl} = 0.001$ shows that both solvers always converge. Unfortunately, the use of such a small time step represents a prohibitive computational approach. We have not investigated what is the maximum time step allowed (between 1 and 30 min at 40-km resolution) for the solvers to converge in all cases.

To illustrate the high convergence rate of the JFNK method as opposed to the ones of Stand-cap and Stand-tanh, Fig. 5 shows the residual norm of the nonlinear system of equations as a function of the iteration (Newton iteration or OL iteration) down to a small residual norm $10^{-6}$. This typical result is for 1 January 1990 18Z. The Stand-cap solver needs in this case 2631 OL iterations to reach a residual norm of $10^{-6}$ while it takes 24 Newton iterations for JFNK to satisfy the same criterion. This might suggest that JFNK is more than a 100 times faster than the Stand-cap solver. This is however not the case because one JFNK iteration involves more calculation (in the fast phase) than one OL iteration. JFNK is 23 times faster than the Stand-cap solver to reach a residual norm of $10^{-6}$. Compared to the Stand-tanh solver, JFNK is 6.4 times faster. The required CPU time for JFNK is 2.41 s, 15.49 s for Stand-tanh and 55.07 s for Stand-cap.

Even though the convergence rate of the JFNK solver is high (especially in the fast phase), it is not quadratic because an inexact Newton approach is used. Asymptotic quadratic convergence could be possible but at the expense of very small $c_k$ values [8].

5.2. Discussion about the robustness of the standard and JFNK solvers

Both standard and JFNK solvers show a lack of robustness. Moreover, the failure rate for both solvers increases as the grid is refined. However, the lack of robustness of the solvers might not be so dramatic for practical considerations. First,
Does it matter?

Losch et al. (2014)
solution method: EVP variants

\[
\sigma_{ij} = \frac{P}{2\Delta} \left\{ 2\ddot{e}_{ij}e^{-2} + \left[ (1 - e^{-2})(\dot{e}_{11} + \dot{e}_{22}) - \Delta \right] \delta_{ij} \right\}
\]

\[
\Leftrightarrow \quad \frac{\Delta e^2}{P} \sigma_{ij} + \left[ \frac{\Delta(1 - e^2)}{2P}(\sigma_{11} + \sigma_{22}) + \frac{\Delta}{2} \right] \delta_{ij} = \dot{e}_{ij}
\]

• Hunke and Dukowicz (1997)
• does not converge (definitely not to VP, Lemieux et al. 2012, Losch and Danilov 2012)
• adding inertial term to momentum equations fixes convergence (Lemieux et al. 2012, Bouillon et al 2013)
• m(odified)EVP, a(daptive)EVP (Kimmritz et al 2015, 2016, 2017)
issues with conventional EVP

reference

EVP, 120 sub-cycles

EVP, 1800 sub-cycles

Lemieux et al. (2012), shear and divergence (per day)
New EVP equations

based on Lemieux et al. (2012), Bouillon et al. (2013), add “inertial-like” term to momentum equations

\[
\begin{align*}
\sigma^{p+1} - \sigma^p &= \frac{1}{\alpha} \left( \sigma(u^p) - \sigma^p \right), \\
u^{p+1} - u^p &= \frac{1}{\beta} \left( \frac{\Delta t}{m} \nabla \cdot \sigma^{p+1} + \frac{\Delta t}{m} R^{p+1/2} \right)
\end{align*}
\]
New EVP equations

New momentum equations

\[
\begin{align*}
\sigma^{p+1} - \sigma^p &= \frac{1}{\alpha} \left( \sigma(u^p) - \sigma^p \right), \\
u^{p+1} - u^p &= \frac{1}{\beta} \left( \frac{\Delta t}{m} \nabla \cdot \sigma^{p+1} + \frac{\Delta t}{m} R^{p+1/2}_\tau + u_n - \alpha \right)
\end{align*}
\]

with 
\( \alpha \beta \gg \gamma = \frac{P}{2\Delta} \frac{(c\pi)^2}{A} \frac{\Delta t}{m} \)

from stability analysis (Kimmritz et al, 2015, 2016).

modified EVP: \( \alpha, \beta = \text{constant, order}(300) \)

adaptive EVP: \( \alpha = \beta = (4\gamma)^{1/2} \)
high resolution simulations
EVP “convergence” (in FESOM)

Koldunov et al. (2019), JAMES
~ 4 km
EVP “convergence” (in FESOM)

Koldunov et al. (2019), JAMES, grid resolution ~ 4 km
Convergence to VP solution:

ice thickness (m) at 4.5 km grid spacing

\[ \alpha \beta \gg \gamma = \zeta \frac{(c\pi)^2 \Delta t}{A_c} \frac{\Delta t}{m} \]

stability parameter depends on grid spacing and local ice viscosity
sea ice dynamics from optical satellite images
Menemenlis, pers. comm
Maxwell Elasto-Brittle rheology

• VP has been criticized for low intermittency and heterogeneity (Girard et al 2009, but Hutter et al 2018/2019 show opposite)
• Dansereau et al (2016):
  - new model variable: damage (actually integrity of sea ice), affects ice strength
  - Mohr-Coulomb law for failure (increases damage) \( (\tau = \mu \sigma + c) \)
  - VP-viscosities are re-interpreted as constant coefficients; leads to a linear problem
• unclear: optimal solution strategy, computational cost (probably very high), stability, coupling to thermodynamics
not sure if I should show this
Biogeochemistry in sea ice models

• ice BGC affects attenuation (uptake) of shortwave radiation (albedo; modifies melt rate and availability to ocean), ocean BGC through “seeding” with biologically active material
• simple models for simple sea ice models, e.g. SIMBA (Castellani et al. 2017)
• multilayer models require more sophisticated sea ice models with positive definite vertical transport schemes, see documentation of “icepack” (column physics and biogeochemistry of CICE)
• numerous feedbacks need to be taken into account
What’s missing and where to go from here

• dynamics
  - explore continuity assumption at high resolution
  - new rheological approaches, anisotropy (EAP)
  - discrete element models for climate research?
  - surface and bottom stress (skin drag, form drag, ice roughness length, etc.)
  - porosity in ridges => ice strength parameterisations (Roberts et al. 2019)
• thermodynamics (most of this is in CICE/ICEPACK: https://github.com/CICE-Consortium)
  - multiple layers, vertical advection
  - melt pond parameterisations
  - snow parameterisations
  - biogeochemistry
• coupling to ocean and atmosphere
• …
Literature about sea ice modelling

- Reviews: